SEPARATE HOLOMORPHIC EXTENSION ALONG LINES AND HOLOMORPHIC EXTENSION FROM THE SPHERE TO THE BALL

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ABSTRACT. We give positive answer to a conjecture by Agranovsky. A continuous function on the sphere which has separate holomorphic extension along the complex lines which pass through three non aligned interior points, is the trace of a holomorphic function in the ball.
MSC: 32F10, 32F20, 32N15, 32T25

1. Introduction

The problem of describing families of discs which suffice for testing analytic extension of a function f from the sphere $\partial\mathbb{B}^2$ to the ball $\mathbb{B}^2$ has a long history. For f continuous on $\partial\mathbb{B}^2$, Agranovsky-Valski [4] use all the lines, Agranovki-Semenov [3] the lines through an open subset $D' \subset \mathbb{B}^2$, Rudin [10] the lines tangent to a concentric subsphere $B^2_{\frac{1}{2}}$, Baracco–Tumanov-Zampieri the lines tangent to any strictly convex subset $D' \subset\subset \mathbb{B}^2$. There are many other contributions such as [2],[11], [8] just to mention a few. It is a challenging attempt to reduce the number of parameters in the testing families. However, one encounters an immediate constraint: lines which meet a single point $z_o \in \mathbb{B}^2$ do not suffice. Instead, two interior points or a single boundary point suffice: Agranovsky [1] and Baracco [5]. However, in these last two results, the reduction of the testing families is compensated by an assumption of extra initial regularity: f is assumed to be real analytic. Globevnik [7] shows that, for two points, C^∞-regularity still suffices, but C^k does not. This suggests that holomorphic extension is a good balance between reduction of testing families and improvement of initial regularity. And in fact, it is showed here, that for $f \in C^0$ three not on the same line points suffice. Here is our result.

Theorem 1.1. *Let f be a continuous function on the sphere $\partial\mathbb{B}^2$ which extends holomorphically along any complex line in $\mathbb{B}^2$ which encounters the set consisting of 3 points not on the same line. Then, f extends holomorphically to $\mathbb{B}^2$.*

The proof follows in Section 2 below. It shows that, the result should hold for a ball of general dimension $\mathbb{B}^n$. In this case, $n+1$ points in generic position should suffice. We first introduce some terminology. A straight disc A is the intersection of a straight complex line with $\mathbb{B}^2$; $\mathbb{P}T^*\mathbb{C}^2$ is the cotangent bundle with projectivized fibers, and π the projection on the base space; $\mathbb{P}T^*_{\partial\mathbb{B}^2}\mathbb{C}^2$ the projectivized conormal bundle to $\partial\mathbb{B}^2$ in $\mathbb{C}^2$. It is readily seen that the straight discs A of the ball are the geodesics of the Kobayashi metric, or, equivalently, the so called "stationary discs" (cf. Lempert [9]). These are the discs endowed with a meromorphic lift $A^* \subset \mathbb{P}T^*\mathbb{C}^2$ with a simple pole attached to $T^*_{\partial\mathbb{B}^2}\mathbb{C}^2$, that is, satisfying $\partial A^* \subset \mathbb{P}T^*_{\partial\mathbb{B}^2}\mathbb{C}^2$. We fix three points P_j, $j = 1, 2, 3$ in $\mathbb{B}^2$ and consider a set, indexed by j, of (2)-parameter families of straight discs A^j passing through P_j. We define M_j to be the union of the lifts of the family with index j. The set M_j is generically a CR manifold with CR dimension 1 except at the points that project over P_j; we denote by M_j^{reg} the complement of this set. The boundary of M_j concides with $\mathbb{P}T^*_{\partial\mathbb{B}^2}\mathbb{C}^2$ which is maximal totally real in $\mathbb{P}T^*\mathbb{C}^2$. Here is the central point of our construction. Though the function f, in the beginning of the proof, is not extendible to $\mathbb{B}^2$ as a result of the separate extensions to the A's, nevertheless it is naturally lifted to a function F on M_j by gluying the bunch of separate holomorphic extensions to the lifts A^*'s. This is defined by

$$F(z, [\zeta]) = f_{A_{(z,[\zeta])}}(z),$$

where $A_{(z,[\zeta])}$ is the unique stationary disc whose lift $A^*_{(z,[\zeta])}$ passes through $(z, [\zeta])$. The crucial point here is that the A's may overlap on $\mathbb{C}^2$ but the A^*'s do not in $\mathbb{P}T^*\mathbb{C}^2$. The function F is therefore well defined and CR on M_j^{reg}.

2. Proof of Theorem 1.1

The proof consists of several steps. We start by collecting some easy computations. We identify $\mathbb{P}T^*\mathbb{C}^2 \simeq \mathbb{C}^2 \times \mathbb{CP}_1 \simeq \mathbb{C}^3$ with coordinates $(z_1, z_2) \in \mathbb{C}^2$ and $z_3 = \frac{\zeta_2}{\zeta 1} \in \mathbb{CP}_1$. Let M_0 be the collection of the lifts of the discs through 0.

Lemma 2.1. *Let A_0^* be the (unique) disc of M_0 which projects over the z_1-axis. Then, A_0^*, identified to a disc of $\mathbb{C}^3$, has two holomorphic lifts to $T^*\mathbb{C}^3$ attached to $T^*_{M_0}\mathbb{C}^3$. Their components are parametrized by $z_1 \mapsto (0, -\frac{1}{z_1}, 1)$ and $z_1 \mapsto (0, \frac{1}{iz_1}, \frac{1}{i})$respectively.*

Proof. First, we notice that for any $z = (z_1, z_2) \in \mathbb{B}^2$ the disc $\tau \mapsto \tau\frac{z}{\|z\|}$ is the only passing through z and 0. The lift attached to the

projectivized conormal bundle of this disc is the constant $[\bar{z}]$. We have

$$M_0 = \{(z; [\bar{z}])\ z \in \mathbb{B}^2 \setminus 0\} \cup \{(0; [\zeta])\ \forall [\zeta] \in \mathbb{CP}_1\}.$$

Clearly M_0 (or more precisely M_0^{reg}) has equation $r : z_3 - \frac{\bar{z}_2}{\bar{z}_1} = 0$. In particular the lift of A_0 to $\mathbb{P}T^*\mathbb{C}^2$ is $A_0^*(\tau) = ((\tau, 0); [1, 0])$ which in coordinates is expressed by $A_0^*(\tau) = (\tau, 0, 0)$. Since M_0 is Levi flat, the space of holomorphic lifts contained in T^*M_0 has dimension two. For instance a basis for the space of lifts is given by

(2.1)

$$\omega_1(z_1, z_2) = \partial \mathsf{Re}\, r = \left(\frac{z_2}{z_1^2}, -\frac{1}{z_1}, 1\right) \text{ and } \omega_2(z_1, z_2) = \partial \mathsf{Im}\, r = \frac{1}{i}\left(-\frac{z_2}{z_1^2}, \frac{1}{z_1}, 1\right).$$

In particular, along A_0^* the conormal bundle to M_0 is generated by $\omega_1(z_1, 0) = (0, \frac{-1}{z_1}, 1)$ and $\omega_2(z_1, 0) = (0, \frac{1}{iz_1}, \frac{1}{i})$. As one can readily note both sections are holomorphic along A_0^* and they are exactly the lifts of A_0^* to the conormal bundle of $T^*_{M_0}\mathbb{C}^3$.

□

Remark 2.2. Note that if in the lemma above we consider the union of the lifts of discs passing through the point $P_{\zeta_0} = (\zeta_0, 0)$ the manifold resulting M_{ζ_0} still contains A_0^* (i.e. the z_1 axis) and along the boundary of A_0^* we have $TM_0|_{\partial A_0^*} = TM_{\zeta_0}|_{\partial A_0^*}$ and thus also $T^*_{M_0}\mathbb{C}^3|_{\partial A_0^*} = T^*_{M_{\zeta_0}}\mathbb{C}^3|_{\partial A_0^*}$. From this equality we have that if $\tilde{\omega}_1$, $\tilde{\omega}_2$ is a basis of lifts of A_0^* to the conormal bundle to M_{ζ_0}, then this is related to the basis ω_1, ω_2 by

$$\mathrm{Span}\{\tilde{\omega}_1, \tilde{\omega}_2\}|_{\partial A_0^*} = \mathrm{Span}\{\omega_1, \omega_2\}|_{\partial A_0^*}. \tag{2.2}$$

Combination of (2.2) with the fact that singularity of $\tilde{\omega_1}, \tilde{\omega_2}$ must now be located at ζ_0 yields a choice of holomorphic basis as $\tilde{\omega_1}(z_1) = \left(0, -\frac{1}{(z_1-\zeta_0)}, \frac{1}{(1-z_1\bar{\zeta}_0)}\right)$ and $\tilde{\omega_2}(z_1) = \left(0, \frac{1}{i(z_1-\zeta_0)}, \frac{1}{i(1-z_1\bar{\zeta}_0)}\right)$.

Before the proof of our main theorem we need a preliminary crucial result

Theorem 2.3. *Let $P_1, P_2 \in \mathbb{B}^2$ be two distinct points inside the ball and let $f : \partial\mathbb{B}^2 \to \mathbb{C}$ be a continuous function such that f extends holomorphically along every complex straight line passing through either P_1 or P_2. Then for any such disc A, except the one passing through both points, the lifted function F extends holomorphically to a neighborhood of $A^* \setminus \pi^{-1}(P_j)$ where j is 1 or 2 according to $P_1 \in A$ or $P_2 \in A$.*

Proof. It is not restrictive to assume that the disc A is the z_1 axis, that $P_2 = (0, z_2)$ and that $P_1 = (\zeta_0, 0)$. We note that M_1 and M_2 intersect transversally along the boundary of A^*. Let $P = (\zeta, 0)$ be a point of

the boundary of A and $P^*=(\zeta,0,0)$ be the corresponding point on A^*. P^* lies in the common boundary of M_1 and M_2. Let v_ζ be a tangent vector to $T_{P^*}M_2\setminus T_{P^*}E$ which points inside M_2. The equivalence class $[v_\zeta]$ in the vector spaces quotient $\frac{T_{P^*}\mathbb{C}^3}{T_{P^*}M_1}$ is called the pointing direction of M_2 with respect to M_1. We say in this case that F extends at P^* in direction $[v_\zeta]$. Let $Q^*=(\zeta_Q,0,0)$ be a point of A^* ($\zeta_Q\neq\zeta_0$). Following [13] by effect of the extension of F at P^* in direction $[v_\zeta]$ we have extension at Q^* in direction $[w_\zeta]\in\frac{T_{Q^*}\mathbb{C}^3}{T_{Q^*}M_1}$. The relation of $[w_\zeta]$ with the initial $[v_\zeta]$ is expressed by means of contraction with the holomorphic basis of lifts $\tilde{\omega_1},\tilde{\omega_2}$:

(2.3)

$$\mathsf{Re}\,\langle\tilde{\omega}_1(\zeta),v_\zeta\rangle=\mathsf{Re}\,\langle\tilde{\omega}_1(\zeta_Q),w_\zeta\rangle \text{ and } \mathsf{Re}\,\langle\tilde{\omega}_2(\zeta),v_\zeta\rangle=\mathsf{Re}\,\langle\tilde{\omega}_2(\zeta_Q),w_\zeta\rangle.$$

In other words the directions of CR extendibility, which are vectors in the normal bundle $\frac{T\mathbb{C}^3}{TM_1}$, are constant in the system dual to $\{\tilde{\omega}_1,\tilde{\omega}_2\}$.

We first compute the pointing direction of M_2 at the point P^*. To this end we first compute the disc passing through P_2 and P which is

$$A_{P_2P}(\tau)=(\frac{|z_2|^2\zeta}{1+|z_2|^2},\frac{z_2}{1+|z_2|^2})+\frac{\tau}{1+|z_2|^2}(\zeta,-z_2);$$

note that $A_{P_2P}(1)=P$. The lift component of A_{P_2P} is

$$A^*_{P_2P}=[|z_2|^2\bar{\zeta}\tau+\bar{\zeta},\bar{z}_2\tau-\bar{z}_2],$$

and dividing the second component by the first we get that the $A^*_{PP_2}$'s coordinates in $\mathbb{C}^3$ are

$$\left((\frac{|z_2|^2\zeta}{1+|z_2|^2}+\frac{\tau}{1+|z_2|^2}\zeta,\frac{z_2}{1+|z_2|^2}-\frac{\tau z_2}{1+|z_2|^2},\frac{\bar{z}_2(\tau-1)}{\bar{\zeta}(|z_2|^2\tau+1)}\right).$$

The pointing direction of M_2 at P is

$$v_\zeta=-\partial_\tau A^*_{P_2P}(1)=\frac{-1}{1+|z_2|^2}(\zeta,-z_2,\frac{\bar{z}_2}{\bar{\zeta}}).$$

We have

$$\mathsf{Re}\,\langle\tilde{\omega}_1(\zeta),v_\zeta\rangle=\frac{-1}{1+|z_2|^2}\mathsf{Re}\,\frac{z_2}{\zeta-\zeta_0} \tag{2.4}$$

and

$$\mathsf{Re}\,\langle\tilde{\omega}_2(\zeta),v_\zeta\rangle=\frac{-1}{1+|z_2|^2}\mathsf{Im}\,\frac{z_2}{\zeta-\zeta_0}. \tag{2.5}$$

The first members of (2.4) and (2.5) express the components in the normal bundle to M_1 of w_ζ with respect to the dual basis of $\omega_1(\zeta_Q),\omega_2(\zeta_Q)$. By letting ζ vary in ∂A we see that $[w_\zeta]$ sweeps all the directions in

$\frac{T\mathbb{C}^3}{TM_1}|_{Q^*}$. Therefore, by the edge of the wedge theorem, F extends holomorphically to a neighborhood of Q^* and, by propagation, to a neighborhood of any other point of A^* except from the point over P_1 where the CR singularity is located.

$\square$

We are ready for the proof of Theorem 1.1

End of Proof of Theorem 1.1 Let A_0 be the disc passing through P_1 and P_3. Then in particular $P_2 \notin A_0$. Applying the theorem above we get that F extends holomorphically to a neighborhood of $A_0^* \setminus \{P_1\}$. By repeating the same argument we see that F extends to a neighborhood of $A_0^* \setminus \{P_3\}$. Therefore F extends to a full neighborhood of A_0^*. For any other straight line A through P_1 we can say that F extends holomorphically to a neighborhood of $A^* \setminus P_1$. By applying the continuity principle to the family of discs formed by A_0^* and all the discs through P_1, we get that F extends holomorphically to a set of the form $V \times \mathbb{CP}^1_{\mathbb{C}}$ where V is a neighborhood of P_1. Therefore F does not depend on the second argument and it is therefore naturally defined on the projection of the collection of all the A^*'s, that is, on $\mathbb{B}^2$.

$\square$

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